Group A

1. Let $f : \Re \to \Re$ be a continuously differentiable function which has at least three distinct zeros. (We say $x$ is a zero of $f$ if $f(x) = 0$). Let $g : \Re \to \Re$ be defined as follows: $g(x) = e^{x/2} f(x)$ for all $x \in \Re$.

   (i) Prove that $g$ has at least three distinct zeros.

   (ii) Prove that the function $f + 2f'$ has at least two distinct zeros.

(10+20=30)

2. (i) Consider the following two variable optimization problem:

$$\max_{x,y} \ (x^2 + y^2)$$

$$\text{subject to}$$

$$x + y \leq 1$$

$$x, y \geq 0$$

   Find all solutions of this optimization problem.

   (ii) In a kingdom far, far away, a King is in the habit of inviting 1000 senators to his annual party. As a tradition, each senator brings the King a bottle of wine. One year, the Queen discovers that one of the senators is trying to assassinate the King by giving him a bottle of poisoned wine. Unfortunately, they do not know which senator, nor which bottle of wine is poisoned, and the poison is completely indiscernible. However, the King has 10 prisoners he plans to execute. He decides to use them as taste testers to determine which bottle of wine contains the poison. The poison when taken has no effect on the prisoner until exactly 24 hours later when the infected prisoner suddenly dies. The King needs to determine which bottle of wine is poisoned by tomorrow so that the festivities can continue as planned. Hence he only has time for one round of testing. How can the King administer the wine to the prisoners to ensure that 24 hours from now he is guaranteed to have found the poisoned wine bottle?

(12+18=30)

3. (i) Let $A$ and $B$ be matrices for which the product $AB$ is defined. Show that if the columns of $B$ are linearly dependent, then the columns of $AB$ are linearly dependent.

(ii) Let $e_i$ denote the column vector with three elements, each of which is zero, except for the $i$-th element, which is 1. Consider a linear transformation $L : \Re^3 \to \Re^3$ with $L(e_1) = e_1$, $L(e_2) = e_1 + e_2$, and $L(e_3) = e_2 + e_3$. Does $L$ map $\Re^3$ onto $\Re^3$? Prove your answer.

(15+15=30)

Group B

4. This question pertains to a situation in which a particular commodity, like rice, is both available at a subsidised rate from a fair price shop (ration shop) and at a higher price from the open market. Suppose a consumer can buy a certain (fixed) quantity of rice at a lower price from the ration shop (that is, there is a ration quota). In addition, he can buy more of rice (assume a uniform quality of rice) from the open market at a higher price. (You may assume that consumers preferences are represented by standard downward sloping, smooth, convex indifference curves.)

   (i) Graphically depict the consumer's equilibrium (assuming he exhausts the ration quota and in addition buys from the open market).

   (ii) Suppose rice is a normal good. What will happen to the quantity of rice purchased from the open market (over and above the ration quota) in equilibrium if there is a cut in the ration quota? Briefly explain.

   (iii) Suppose rice is a normal good. What will happen to the quantity purchased in the open market (over and above the ration quota) if the subsidised price (price at which the ration quota rice could be bought) is increased (but is still lower than the open market price)? Will your conclusion change if rice is an inferior good? Briefly explain.

(10+10+10=30)

5. There are plenty of fish in the Dull Lake. Boats can be hired by fishermen to catch fish and sell it on the fish market. The revenue earned each month from a total of x boats is given by the following expression: Rupees $10,000\{4x - \frac{1}{2}x^2\}$. Each boat costs Rupees 20,000 each month.

   (i) Derive the marginal and average revenue per boat

   (ii) The Dull municipality is considering giving out permits for each boat that fishes so they can track who is fishing from the lake. If these permits are allocated freely, how many boats will fish every month?

   (iii) If total profit is to be the maximum possible, how many boats should fish every month?

   (iv) Dull municipality decides to charge for the permits instead of giving them out for free. What should the per-boat charge for the permit be if total profits are to be the maximum possible?

(5+10+5+10=30)

6. The Shoddy Theater screens movies every week and is located on a university campus which has only students and faculty as residents. It is the only source of watching movies for both faculty and students, and is large enough to accomodate all faculty and students. Faculty demand for movie tickets is given by $500 - 4P_F = Q_F$, where $P_F$ refers to the price of the ticket paid by faculty and $Q_F$ refers to the number of tickets purchased by faculty. Demand by students is described by $100 - 2P_S = Q_S$, where $P_S$ refers to the price of the ticket paid by students and $Q_S$ refers to the number of tickets purchased by students. The cost to service demand equals 500.

   (i) If the price charged is to be the same for faculty and students, what price would Shoddy Theater set in order to maximize its profits?

   (ii) Now imagine that Shoddy Theater decides to charge different prices for faculty and students. What would these prices be, if Shoddy Theater wants to maximize profits?

(15+15=30)

Group C

7. Consider a Solow type economy, producing a single good, according to the production function:

$$Y(t) = K(t)^{\alpha} \, L(t)^{(1-\alpha)}$$

Where, $0 < \alpha < 1$, and $Y(t), K(t)$ and $L(t)$ are the output of the good, input of capital, and input of labour used in the production of the good, respectively, at time t. Capital and labour are all fully employed.

Labour force grows at the exogenous rate, $\eta > 0$, i.e.,

$$\frac{1}{L(t)} \frac{dL(t)}{dt} = \eta > 0.$$

Part of the output is consumed and part saved. Let $0 < s < 1$ be the fraction of output that is saved and invested to build up the capital stock. Also assume that there is no depreciation of capital stock.

Then it follows that:

$$sY(t) = \frac{dK(t)}{dt}$$

Where $\frac{d}{dt}$ is the time derivative.

With this above given description of the economy, one can find out the steady state growth rate of Y, for this economy. Growth rate of output is given by: $\frac{1}{Y(t)} \frac{dY(t)}{dt} \equiv g_Y$.

Assume, the economy begins at date 0, from a per capita capital stock, $k(0) \equiv \frac{K(0)}{L(0)} < k^*$, where $k^*$ denotes the steady state per capita capital stock.

(i) Demonstrate formally, whether the growth rate of output $(g_Y)$, at the beginning date 0, is greater, equal or less than the steady state growth rate of output.

(ii) For the same economy, consider, two alternative beginning date scenarios: with per capita capital stock, given by:
Case 1. $k(0)$; Case 2. $k'(0)$. Where, $k(0) < k'(0) < k^*$ Can you compare the beginning date growth rates of output in the two cases?

(iii) Next, consider two Solow type economies, namely, A and B. They are isolated from each other and are working on their own. Both the economies have absolutely the same description as given before, except for the fact that the fraction of income saved in country A, denoted by $s_A$ is greater than the fraction of income saved in country B, denoted by $s_B$. Let $k^A(0)$ be the initial date per capita capital stock in country A and $k^B(0)$, the initial date per capita capital stock in country B, which are both less than their respective steady state values. Assume, $K^A(0) < K^B(0)$. Can you figure out whether the initial date growth rate of output in country A is greater than, equal or less than the initial date growth rate of output in country B? In case you find the data provided to you is insufficient to make any comment on this, please point it out.

(20+5+5=30)

8. (i) What is the money multiplier? What determines its size? What is the relation between the monetary base, the money multiplier, and the money supply? Which of these variables can the central bank change to change the money supply? What is the direction of change in each case?

(ii) Why might the cash/deposit ratio and the reserve to asset ratio be decreasing functions of the rate of interest? How does an interest-sensitive money supply affect the LM curve? Illustrate with a diagram, comparing this LM curve with the standard LM. How does this change the effectiveness of counter-cyclical fiscal policy (in a closed economy)? Explain.

(15+15=30)

9. (i) What is the difference between the real and the nominal exchange rate? Give an example to explain this to someone who has not studied economics. Is an increase in the real cost of imports an improvement or a deterioration in the terms of trade?

(ii) A small open economy has a government budget surplus and a trade deficit. Explain whether there is a private sector surplus, deficit or balance. Examine the consequences in the short run for output, the trade balance and the government budget balance of a sudden fall in private consumption in this economy (due to an epidemic in the small country) under (a) fixed exchange rates, (b) flexible exchange rates. Use the Mundell-Fleming model with perfect capital mobility. Explain the adjustment mechanisms.

(10+20=30)