1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that there always exists a point $c > 1$ such that $f(2) - 2f(1) = cf'(c) - f(c)$. [4]

   (b) Let $f : (a, b) \to \mathbb{R}$ be a function such that there is some $c \in (a, b)$ satisfying $f(c) = \max\{f(x) : x \in (a, b)\}$, where $(a, b) = \{x \in \mathbb{R} : a < x < b\}$. Assume that one-sided derivatives $f'_-(c)$ and $f'_+(c)$ exist. Show that $f'_-(c) \geq 0$ and $f'_+(c) \leq 0$. [8]

   (c) Let f be a function which is continuous on $[0, 1]$ and differentiable on $(0, 1)$, with $f(0) = f(1) = 0$. Assume that there is some $c \in (0, 1)$ such that $f(c) = 1$. Prove that there exists some $x_0 \in (0, 1)$ such that $|f'(x_0)| > 2$. [12]

2. (a) A restaurant has a menu card containing three Chinese items, six Indian items and five Continental items. Six items are selected at random. Let X and Y denote respectively the number of Indian and Continental items selected. Compute the conditional probability mass function of X given that $Y = 3$. Also compute $E(X|Y = 3)$. [8+4=12]

   (b) Let the joint probability mass function of X and Y be

$$p(1,1) = 1/9,\ p(2,1) = 1/3,\ p(3,1) = 1/9,$$

$$p(1,2) = 1/9,\ p(2,2) = 0,\ p(3,2) = 1/18,$$

$$p(1,3) = 0,\ p(2,3) = 1/6,\ p(3,3) = 1/9,$$

   where $p(x, y) = P(X = x, Y = y)$ for all $x = 1, 2, 3$ and $y = 1, 2, 3$. Find the correlation between X and Y. Also find $E(X|Y = 3)$. [8+4=12]

3. A consumer consumes two goods: visits to a nearby park (x) and a composite consumption good (y), according to preferences $u = xy$. The consumer's income is R and the price of the composite good is normalized to 1.
Initially, there is an entry fee p^* per park visit. Suppose the authorities are considering a proposal to reduce the per visit entry fee from p^* to p'.

   (a) Find the increase in consumer's surplus resulting from this proposal. [3]

   (b) Now suppose there is an alternative proposal to maintain the per visit entry fee at the initial level p^*, but hand out a *one-off cash* voucher to the consumer (i.e., the consumer would get the cash voucher only once, regardless of how many times she visits the park). Find the minimum value of the cash voucher that would make the consumer not worse off under this alternative proposal, compared to the earlier proposal to reduce the entry fee from p^* to p'. [7]

   (c) Next, suppose there is a third proposal which simultaneously reduces the per visit entry fee from p^* to p', and charges a *one-off lump-sum* user fee (i.e., a fee which has to be paid only once, regardless of the number of visits). Find the maximum value of the lump-sum user fee that would make the consumer not worse off under this third proposal, compared to the initial situation (i.e., price p^* per visit and no lump-sum payment). [7]

   (d) Prove that $C < \Delta S < E$, where ΔS is the solution to part (a), E to part (b) and C to part (c) above. [7]

4. There is a worker who can purchase e units of education, $e \in [0, \infty]$, at cost $2e^2/\theta$. The worker can be of high ability ($\theta = 2$) or low ability ($\theta = 1$), with the worker knowing her own ability (ability is exogenously given). Note that it is less costly for the worker to achieve a particular level of education if she were of high ability than if she were of low ability.
The worker can be hired by a firm paying wage w, and if hired, her marginal product is θ, where θ is her ability. The firm does not know the worker's ability (it knows θ is either 1 or 2), but might be able to infer it after observing her education choice. In particular, if the firm believes the worker is of high ability, it pays wage $w = 2$, while it pays wage $w = 1$ if it believes the worker is of low ability.
The worker chooses e to maximize utility, and if she works for the firm at wage w, then her utility, given e and θ, is $u(w, e|\theta) = w - 2e^2/\theta$.

   (a) Draw the indifference map of the worker in the education-wage space (assume education to be measured along the horizontal axis and wage to be measured along the vertical axis) if she were of high ability. Also draw the indifference map of the worker if she were of low ability. How many times can an indifference curve of a worker who is of low ability intersect an indifference curve of a worker who is of high ability? [2+2+2]

   (b) Suppose the firm perfectly infers the worker's ability upon observing her education choice. Suppose also the worker chooses $e = 0$ if she is of low ability. Then how much education would she purchase if she were of high ability? [9]

   (c) Continue to assume that the firm perfectly infers the worker's ability upon observing her education choice. Suppose, if the worker is of high ability, her education choice matches what you found in part (b). Then show that she would purchase no education if she were of low ability. [9]

5. An individual lives for two periods, 1 and 2, and has lifetime utility function $U(C_1)+\beta U(C_2)$, where C_1 and C_2 are the consumptions of this individual in period 1 and period 2 respectively, and $0 < \beta < 1$ is the subjective discount factor. The following conditions are satisfied: $U'(C) > 0, U''(C) < 0, \lim_{C\to 0} U'(C) = \infty, \lim_{C\to\infty} U'(C) = 0$.
The individual earns an exogenously given income $w > 0$ in period 1 of his life and earns nothing in the second period of his life. But he can lend or borrow freely at an exogenously given rate of interest $r > 0$.
It follows that $C_1 = w - S$ and $C_2 = (1+r)S$, where S is the savings made by the individual in period 1. Note that C_1 (or S) and C_2 are endogenously determined and the exogenously given parameters of the model are w, r and β.

    (a) Suppose $\beta(1+r) = 1$. Solve for C_1 and C_2 in terms of the exogenously given parameters. [4]

    (b) Now remove the restriction that $\beta(1+r) = 1$. How would savings S be affected if β goes up, ceteris paribus? [8]

    (c) Continuing without the restriction that $\beta(1+r) = 1$, show that savings S goes up as w increases, ceteris paribus. Further show that the increase in S is less than the increase in w. [8+4]